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# THE EQUATIONS OF COMPATIBILITY OF DEFORMATIONS†

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Nine equations of compatibility of deformations are obtained in which, unlike the classical Saint-Venant compatibility equations, only first derivatives with respect to the coordinates occur. It is proved that, of these nine equations, only six are independent. It is shown that the classical compatibility equations can be obtained from these equations. © 2002 Elsevier Science Ltd. All rights reserved.

The classical deformation compatibility equations have been discussed in some detail in monographs on the theory of elasticity [1–4]. These equations consist of six second-order partial differential equations in the six components of the strain tensor. In the linear theory of elasticity the deformation compatibility equations are regarded as the conditions for the six differential equations, which connect the components of the displacement vector and of the linear strain tensor with one another, to be integrable.

## 1. THE CLASSICAL DEFORMATION COMPATIBILITY EQUATIONS

We will use a rectangular system of coordinates $x_1$, $x_2$, $x_3$. The classical deformation continuity (compatibility) equations (Saint-Venant's equations) have the form

$$\frac{\partial^2 \varepsilon_{x_1}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{x_2}}{\partial x_1^2} = \frac{\partial^2 \gamma_{x_1 x_2}}{\partial x_1 \partial x_2} \quad (1\ 2\ 3) \tag{1.1}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial \gamma_{x_3 x_1}}{\partial x_2} + \frac{\partial \gamma_{x_1 x_2}}{\partial x_3} - \frac{\partial \gamma_{x_2 x_3}}{\partial x_1}\right) = 2\frac{\partial^2 \varepsilon_{x_1}}{\partial x_2 \partial x_3} \quad (1\ 2\ 3) \tag{1.2}$$

Here and henceforth the symbol (1 2 3) denotes that the unwritten equations (or expressions) are obtained by cyclic permutation of the subscripts, and $\varepsilon_{x1}$, $\gamma_{x1x2}/2$ (1 2 3) are the components of the linear symmetrical strain tensor.

Using a three-dimensional Fourier integral transformation it was shown in [5] that of the six equations (1.1) and (1.2) only three are independent.

Equations (1.1) and (1.2) contain second derivatives of the components of the strain tensor with respect to the coordinates. Below we will obtain deformation compatibility equations which contain only first derivatives.

## 2. THE ASYMMETRICAL STRAIN TENSOR

It is well known that each angular deformation consists of two parts (two angles). Introducing the notation ($u_1$, $u_2$, $u_3$ are the components of the displacement vector)

$$\gamma_{x_2}^{u_1} = \frac{\partial u_1}{\partial x_2} \quad (1\ 2\ 3) \tag{2.1}$$

we have

$$\gamma_{x_1 x_2} = \gamma_{x_2}^{u_1} + \gamma_{x_1}^{u_2} \quad (1\ 2\ 3) \tag{2.2}$$

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The angles $\gamma_{x2}^{u1}$, $\gamma_{x1}^{u2}$ (1 2 3) are shown in Fig. 1.

The basis of this approach is the splitting of each angular deformation into two parts (angles).

From the three linear deformations $\varepsilon_{x1}$ (1 2 3) and six angles $\gamma_{x2}^{u1}$, $\gamma_{x3}^{u1}$ (1 2 3) we set up a new (asymmetrical) tensor

$$\hat{D} = \begin{Vmatrix} \varepsilon_{x_1}^{u_1} & \gamma_{x_2}^{u_1} & \gamma_{x_3}^{u_1} \\ \gamma_{x_1}^{u_2} & \varepsilon_{x_2}^{u_2} & \gamma_{x_3}^{u_2} \\ \gamma_{x_1}^{u_3} & \gamma_{x_2}^{u_3} & \varepsilon_{x_3}^{u_3} \end{Vmatrix} \tag{2.3}$$

Using notation (2.1) we have

$$\varepsilon_{x_1} = \varepsilon_{x_1}^{u_1} \quad (1\ 2\ 3) \tag{2.4}$$

Tensor (2.3) can be obtained from the displacement vector **u** as follows:

$$\hat{D} = (\nabla \mathbf{u})^* = \frac{d\mathbf{u}}{d\mathbf{r}} \tag{2.5}$$

($(\nabla \mathbf{u})^*$ is the derivative of the vector **u** with respect to the radius vector **r** [2]).

We further have

$$\hat{D} + \hat{D}^* = (\nabla \mathbf{u})^* + \nabla \mathbf{u} = 2\,\mathrm{def}\,\mathbf{u} = 2\hat{\varepsilon}$$

where $\hat{\varepsilon}$ is the classical linear strain tensor, i.e. the tensor $\hat{\varepsilon}$ is the symmetrical part of the tensor $\hat{D}$.

Consequently,

$$\hat{D} = \hat{\varepsilon} + \hat{\Omega}, \qquad \hat{\Omega} = (\hat{D} - \hat{D}^*)/2$$

## 3. DERIVATION OF THE NEW DEFORMATION COMPATIBILITY EQUATIONS

We set up the tensor

$$\hat{B} = \mathrm{rot}\,\hat{D}^* = \mathrm{rot}(\nabla \mathbf{u}) = 0 \tag{3.1}$$

Hence follow nine new deformation compatibility equations

$$\frac{\partial \gamma_{x_3}^{u_1}}{\partial x_2} - \frac{\partial \gamma_{x_2}^{u_1}}{\partial x_3} = 0 \quad (1\ 2\ 3) \tag{3.2}$$

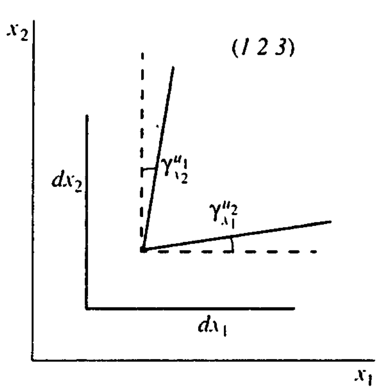


Fig. 1

$$\frac{\partial \varepsilon_{x_1}^{u_1}}{\partial x_3} - \frac{\partial \gamma_{x_3}^{u_1}}{\partial x_1} = 0, \quad \frac{\partial \gamma_{x_2}^{u_1}}{\partial x_1} - \frac{\partial \varepsilon_{x_1}^{u_1}}{\partial x_2} = 0 \quad (1\ 2\ 3) \tag{3.3}$$

Only first derivatives with respect to the coordinates occur in deformation compatibility equations (3.2) and (3.3), whereas second derivatives occur in the classical saint-Venant equations (1.1) and (1.2).

The tensor $\hat{B}$ can be called the deformation compatibility (continuity) tensor.

The classical equations (1.1) and (1.2) can be obtained from the new deformation compatibility equations (3.2) and (3.3).

From Eqs (3.3) we have

$$\frac{\partial \varepsilon_{x_1}^{u_1}}{\partial x_3} = \frac{\partial \gamma_{x_3}^{u_1}}{\partial x_1}, \quad \frac{\partial \varepsilon_{x_3}^{u_3}}{\partial x_1} = \frac{\partial \gamma_{x_1}^{u_3}}{\partial x_3} \tag{3.4}$$

We differentiate the first equation of (3, 4) with respect to $x_3$ and the second with respect to $x_1$ and add them term by term. Using relations (2.2) we obtain

$$\frac{\partial^2 \varepsilon_{x_1}^{u_1}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{x_3}^{u_3}}{\partial x_1^2} = \frac{\partial^2}{\partial x_1 \partial x_3}(\gamma_{x_3}^{u_1} + \gamma_{x_1}^{u_3}) = \frac{\partial^2 \gamma_{x_3 x_1}}{\partial x_1 \partial x_3}$$

i.e. one of Eqs (1.1). The remaining two equations of (1.1) can be obtained similarly.

We will now obtain Eqs (1.2). We have

$$\frac{\partial \gamma_{x_1}^{u_3}}{\partial x_2} + \frac{\partial \gamma_{x_2}^{u_3}}{\partial x_1} = 2\frac{\partial^2 u_3}{\partial x_1 \partial x_2} \tag{3.5}$$

$$\frac{\partial \gamma_{x_3}^{u_2}}{\partial x_1} + \frac{\partial \gamma_{x_3}^{u_1}}{\partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = \frac{\partial \gamma_{x_1 x_2}}{\partial x_3} \tag{3.6}$$

We will write equality (3.6) in the form

$$\frac{\partial}{\partial x_1}(\gamma_{x_3}^{u_2} + \gamma_{x_2}^{u_3}) + \frac{\partial}{\partial x_2}(\gamma_{x_1}^{u_3} + \gamma_{x_3}^{u_1}) = \frac{\partial \gamma_{x_1 x_2}}{\partial x_3} + \frac{\partial \gamma_{x_1}^{u_3}}{\partial x_2} + \frac{\partial \gamma_{x_2}^{u_3}}{\partial x_1}$$

Hence, using (2.2) and (3.5) we have

$$\frac{\partial \gamma_{x_2 x_3}}{\partial x_1} + \frac{\partial \gamma_{x_3 x_1}}{\partial x_2} - \frac{\partial \gamma_{x_1 x_2}}{\partial x_3} = 2\frac{\partial^2 u_3}{\partial x_1 \partial x_2}$$

Differentiating this equation with respect to $x_3$ we arrive at one of the equations (1.2). The remaining two equations of (1.2) can be obtained similarly.

## 4. THE NUMBER OF INDEPENDENT DEFORMATION COMPATIBILITY EQUATIONS

We will show that of the nine deformation compatibility equations (3.2) and (3.3) only six are independent.

Applying to Eqs (3.2) and (3.3) a three-dimensional Fourier integral transformation of the form

$$\bar{f}(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3) e^{i(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)} dx_1 dx_2 dx_3$$

and taking into account the conditions for the deformations to decay at infinity, we obtain the algebraic equations

$$\alpha_2 \bar{\gamma}_{x_3}^{u_1} - \alpha_3 \bar{\gamma}_{x_2}^{u_1} = 0 \quad (1\ 2\ 3) \tag{4.1}$$

$$\alpha_3\bar{\varepsilon}_{x_1}^{u_1} - \alpha_1\bar{\gamma}_{x_3}^{u_1} = 0, \quad \alpha_1\bar{\gamma}_{x_2}^{u_1} - \alpha_2\bar{\varepsilon}_{x_1}^{u_1} = 0 \quad (1\ 2\ 3) \tag{4.2}$$

(in order to simplify the proof we consider an unbounded elastic medium).

From Eqs (4.2) we have

$$\bar{\gamma}_{x_3}^{u_1} = \frac{\alpha_3}{\alpha_1}\bar{\varepsilon}_{x_1}^{u_1}, \qquad \bar{\gamma}_{x_2}^{u_1} = \frac{\alpha_2}{\alpha_1}\bar{\varepsilon}_{x_1}^{u_1}$$

Substituting these expressions into the first equation of (4.1), we obtain an identity. A similar result is obtained for the remaining two equations of (4.1).

Hence, of the nine deformation compatibility equations (3.2) and (3.3) only six, namely Eqs (3.3), are independent.

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